Primjena površinskih integrala

I Izračunavanje površine dijela glateke površi, koja pripada prostoru $\mathbb{R}^3$

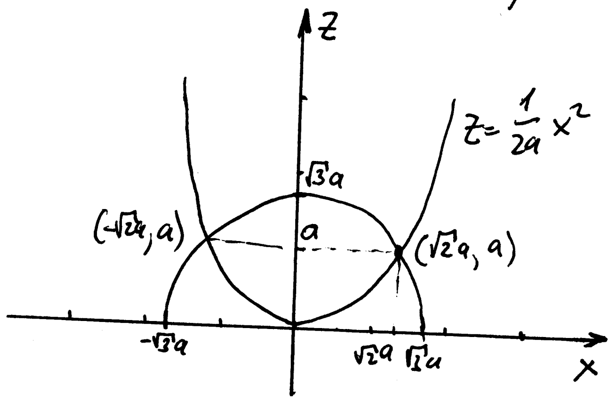
Neka je površ S zadana jednačinom $z = z(x, y)$ gdje su $(x, y) \in D$. (D - je oblast u ravni xOy u koju se projektuje površ $z = z(x, y)$).

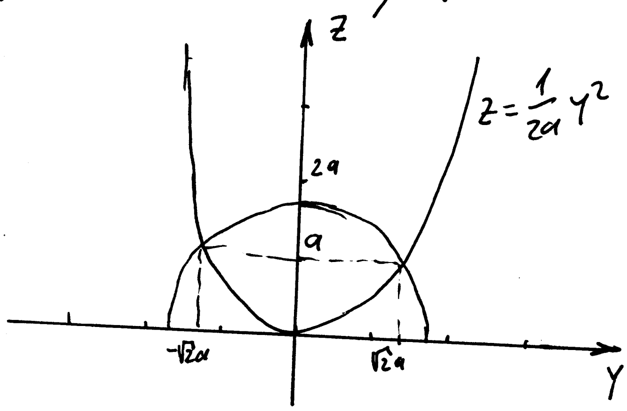
Površina P dijela glatke površi $S \subseteq \mathbb{R}^3$ računa se po formuli

$$P = \iint_S dS = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}\, dx\, dy.$$

\# Izračunati površinu dijela lopte $x^2+y^2+z^2=3a^2$ koja se nalazi ispod parabole $x^2+y^2=2az$ a iznad xOy ravni.

Rj: Na osnovu skica presjeka datih površina sa xOz i yOz ravnima ćemo vidjeti kaka tijela su u pitanju.





$$x^2+z^2=3a^2$$
$$x^2=2az$$
$$z^2+2az-3a^2=0$$
$$D=4a^2+12a^2=16a^2$$
$$z_{1,2}=\frac{-2a\pm 4a}{2}$$
$$z_1=a \quad z_2=-3a$$

$$P=\iint_S dS$$ površinski integral prve ~~vr~~ te

$$z^2=3a^2-x^2-y^2$$
$$z=\pm\sqrt{3a^2-x^2-y^2}$$

U našem slučaju S je $z=\sqrt{3a^2-x^2-y^2}$ i to do ove površine koji se nalazi ispod parabole

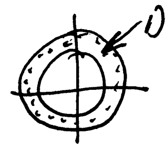
$$P=\iint_S dS=\iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z'_x=\frac{-2x}{2\sqrt{3a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{3a^2-x^2-y^2}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{3a^2-x^2-y^2}+\frac{y^2}{3a^2-x^2-y^2}=\frac{3a^2}{3a^2-x^2-y^2}$$

$$P=\sqrt{3}a\iint_D \frac{dx\,dy}{\sqrt{3a^2-x^2-y^2}}$$

gdje je D projekcija površi S na xOy ravan. U našem slučaju



Uvedimo polarne koordinate

$x = r\cos\varphi$
$y = r\sin\varphi$
$dx\,dy = r\,dr\,d\varphi$

$D \xrightarrow{\text{transformiše}} D': \begin{cases} \sqrt{2}a \le r \le \sqrt{3}a \\ 0 \le \varphi \le 2\pi \end{cases}$

$x^2 + y^2 = r^2$

$$P = \sqrt{3}a \iint\limits_{D'} \frac{r\,dr\,d\varphi}{\sqrt{3a^2 - r^2}} = \sqrt{3}a \int_0^{2\pi} d\varphi \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{r\,dr}{\sqrt{3a^2-r^2}} = \left| \begin{array}{l} 3a^2 - r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\Big|_{\sqrt{2}a}^{\sqrt{3}a} \Rightarrow t\Big|_a^0 \end{array} \right|$$

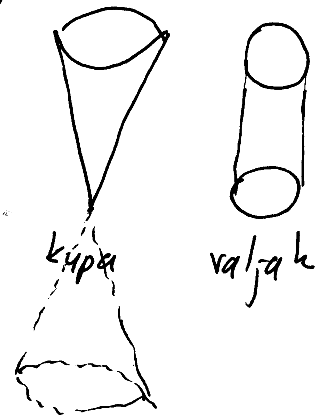
$$= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^a \frac{t\,dt}{t} = 2a^2\sqrt{3}\pi$$

traženo rješenje

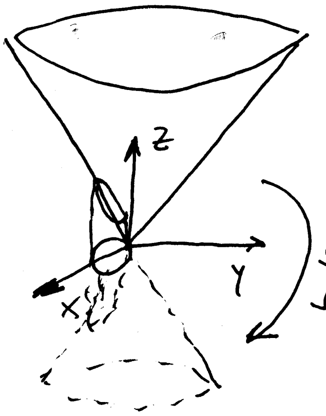
2π a

Izračunati površinu onog dijela kupe $z^2=x^2+y^2$ koji se nalazi unutar valjka $x^2+y^2=2x$.

Rj.



Prema zadatku dio kupe se nalazi unutar valjka



$$x^2+y^2=2x$$
$$x^2-2\cdot x\cdot 1+1+y^2=1$$
$$(x-1)^2+y^2=1$$

sličnu figuru ćemo imati i sa druge strane xOy ravni

$$P=\iint_S dS$$ gdje je S dio kupe koji se nalazi unutar valjka

$$P=\iint_D \sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$$

$$z=\pm\sqrt{x^2+y^2}$$

Ako za z uzmemo $z=\sqrt{x^2+y^2}$ dobićemo površinu dijela kupe iznad xOy ravni.

$$z=\sqrt{x^2+y^2}$$

$$z_x'=\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}},\quad z_y'=\frac{y}{\sqrt{x^2+y^2}},\quad 1+z_x'^2+z_y'^2=1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=1+1=2$$

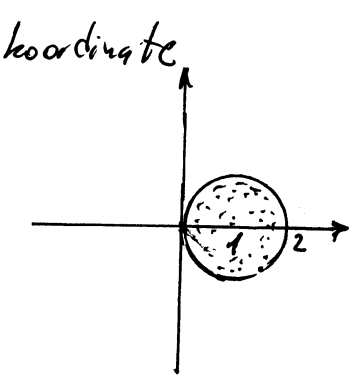
D: unutrašnjost kruga $x^2+y^2=2x$

uvedimo polarne koordinate

$x=r\cos\varphi+1$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$



$$D \xrightarrow{\text{transf.}} D':\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\end{cases}$$

$$\frac{1}{2}P=\iint_D\sqrt{2}\,dx\,dy=\sqrt{2}\iint_{D'} r\,dr\,d\varphi=\sqrt{2}\int_0^{2\pi}d\varphi\int_0^1 r\,dr=\sqrt{2}\cdot\frac{1}{2}r^2\Big|_0^1\,\varphi\Big|_0^{2\pi}=\sqrt{2}\,\pi$$

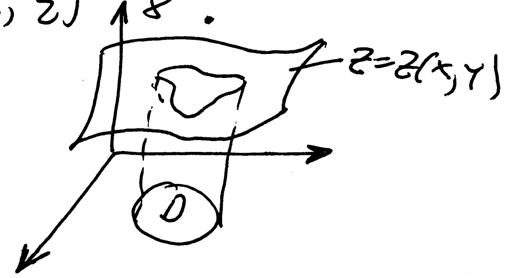
$$P=2\sqrt{2}\,\pi$$

\# Izračunati površinu dijela površi $S: z^2=2xy$ određene u prvom oktantu u presjeku sa ravnima: $x=0$, $y=0$ i $x+y=1$.

Uputa: $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$, $B(\frac{3}{2},\frac{3}{2})=\frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2})=\frac{3\pi}{8}$.

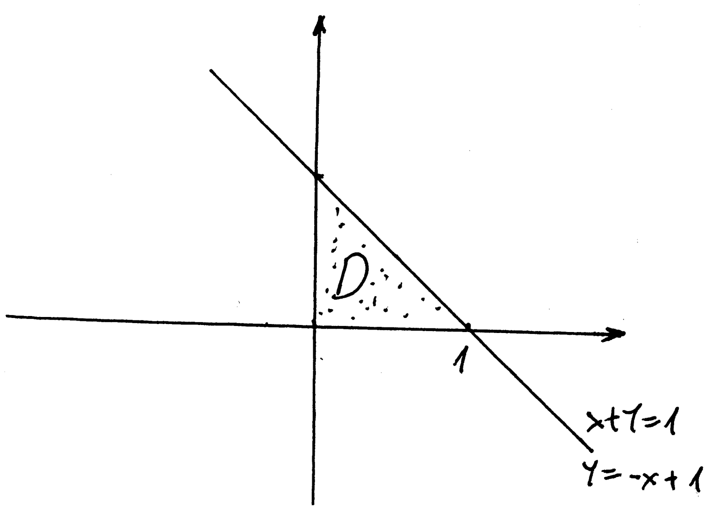
Rj:

$$P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$



Kako je površina S u prvom oktantu, u našem slučaju je

$S:\ z=\sqrt{2}\sqrt{xy}$



$$z'_x=\sqrt{2}\,\frac{y}{2\sqrt{xy}}$$

$$z'_y=\sqrt{2}\,\frac{x}{2\sqrt{xy}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{y^2}{2xy}+\frac{x^2}{2xy}=\frac{2xy+y^2+x^2}{2xy}=\frac{(x+y)^2}{2xy}$$

$$P=\iint_S dS=\iint_D\frac{x+y}{\sqrt{2xy}}\,dx\,dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{-x+1}\frac{(x+y)}{\sqrt{xy}}\,dy=$$

$$=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}+y\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}\right)dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dy$$

$$=\frac{1}{\sqrt{2}}\int_0^1\left(x^{\frac{1}{2}}\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^{1-x}+x^{-\frac{1}{2}}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{1-x}\right)dx=\frac{2}{\sqrt{2}}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx+$$

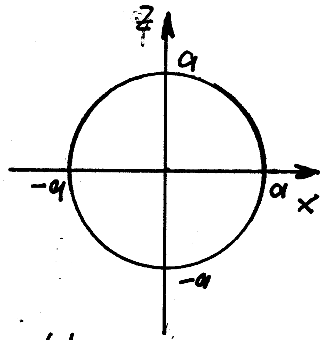
$$+\frac{2}{3\sqrt{2}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}dx=\sqrt{2}\int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}-1}dx+\frac{\sqrt{2}}{3}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{5}{2}-1}dx=\sqrt{2}\,B\left(\frac{3}{2},\frac{3}{2}\right)+$$

$$+\frac{\sqrt{2}}{3}B\left(\frac{1}{2},\frac{5}{2}\right)=\frac{\pi}{2\sqrt{2}}.$$

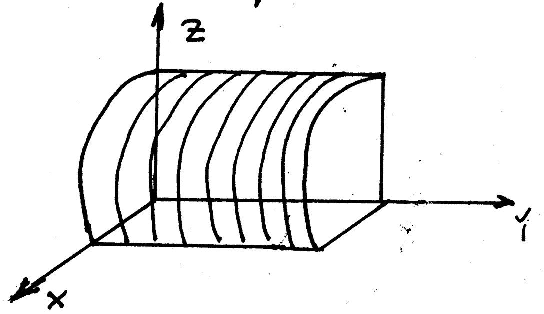
(#) Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+z^2=a^2,\ y \in \mathbb{R}\}$ i $S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid y^2+z^2=a^2,\ x \in \mathbb{R}\}$. Izračunati površinu dobijenog tijela.

Rj. $P = \iint\limits_S dS$ Skicirajmo S_1 i S_2, pa skicirajmo njihov presjek.

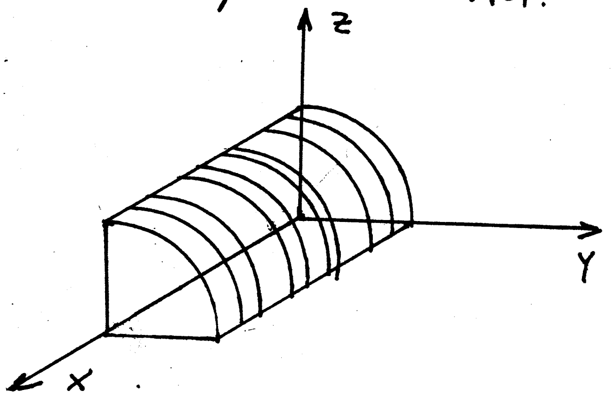
S_1: $x^2+z^2=a^2$ u ravni: xOz



U prostoru, u prvom oktantu:



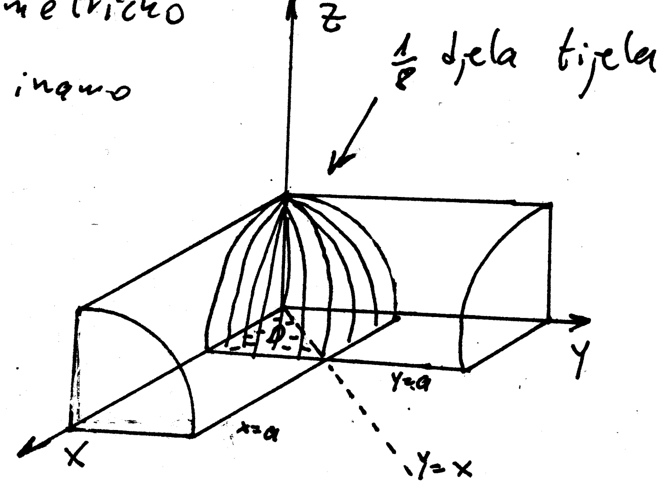
S_2 u prvom oktantu:



Presjek $S_1 \cap S_2$ će kao rezultat dati tijelo koje je simetrično u odnosu na sve tri ravni xOy, xOz i yOz.

$\frac{1}{8}$ dijela tijela će se nalaziti u prvom oktantu:

Primjetimo da je i ovo tijelo simetrično u odnosu na pravu $y=x$ pa imamo



$$P = \frac{1}{16}\iint\limits_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

gdje je $D: \begin{cases} 0 \le x \le a \\ 0 \le y \le x \end{cases}$

$z^2 = a^2 - x^2$ tj. $z = \sqrt{a^2-x^2}$.

$z'_x = \frac{-x}{\sqrt{a^2-x^2}}$, $z'_y = 0$ $\quad 1+(z'_x)^2+(z'_y)^2 = 1+\frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$

tražena površina

$$P = 16a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy = 16a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} = \left| \begin{matrix} a^2-x^2=t \\ -2x\,dx=dt \\ x\,dx=-\frac{1}{2}dt \\ \dots \end{matrix} \right| = \dots = 16a\sqrt{a^2-x^2}\Big|_a^0 = 16a^2$$

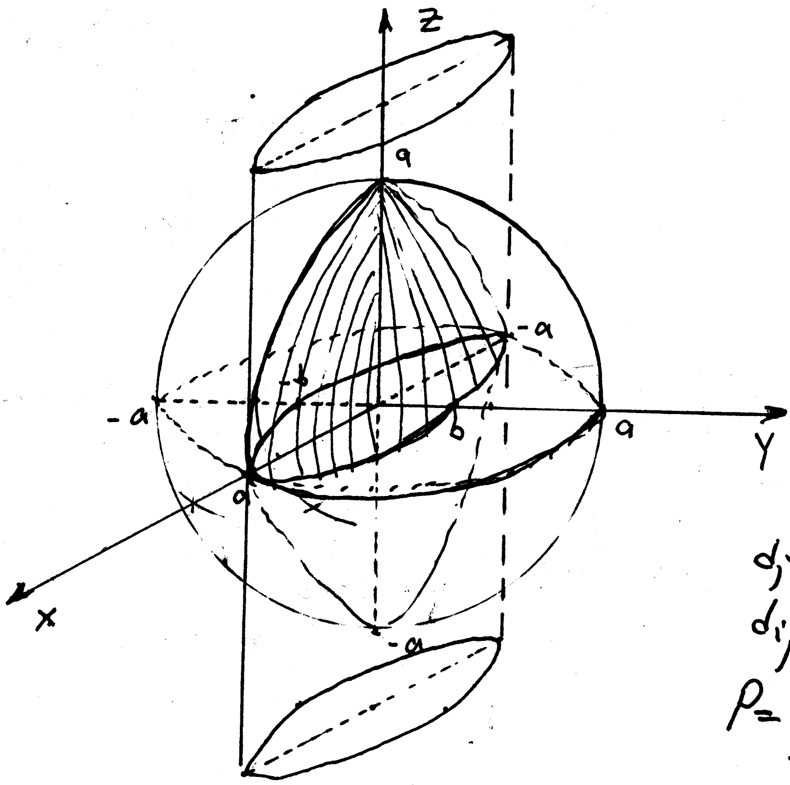
Izračunati površinu dijela sfere

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=a^2\}$$

koji se nalazi u unutrašnjosti cilindra

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z\in\mathbb{R}\},\quad b \le a$$

R: Skicirajmo sferu S i cilindar S_1.



Cilindrična površ u presjeku sa sferom, isjeca iz nje simetričnu površ u odnosu na ravan xOy. Ta dva simetrična dijela označimo sa l_1 i l_2. Svaka od ova dva dijela, koordinatne ravni xOz i yOz ih dijele na četiri jednaka dijela.

$P = \iint\limits_S dS$ gdje je S površina dijela sfere ograničena cilindrom.

$S:\ x^2+y^2+z^2=a^2$

$z = \pm\sqrt{a^2-x^2-y^2}$

Zbog navedene simetričnosti posmatraću sferu samo u prvom oktantu

$z=\sqrt{a^2-x^2-y^2}$ $\quad z'_x = \frac{-x}{\sqrt{a^2-x^2-y^2}},\ z'_y = \frac{-y}{\sqrt{a^2-x^2-y^2}}$

$\iint\limits_S dS = \iint\limits_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$ gdje je D projekcija površi S na xOy ravan

$$1+(z'_x)^2+(z'_y)^2 = 1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2} = \frac{a^2}{a^2-x^2-y^2}$$

$$P = 8\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \\ a\ge 0,\ b\ge 0}} \sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy = 8a\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ a\ge 0,\ b\ge 0} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}} = 8a\iint\limits_D \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

gdje je $D: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \\ x \ge 0,\ y \ge 0 \end{cases}$ ili drugačije napisano $D: \begin{cases} 0 \le x \le a \\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2} \end{cases}$

$$\frac{y^2}{b^2} \le 1-\frac{x^2}{a^2}$$

$$y^2 \le \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$$P = 8a\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{dy}{\sqrt{\underbrace{a^2-x^2}_{\text{ovo je broj za } dy}-y^2}} = 8a\int_0^a \left(\arcsin \frac{y}{\sqrt{a^2-x^2}} \Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}\right) dx$$

$$\left[\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C\right]$$

$$= 8a\int_0^a \left(\arcsin\frac{b}{a} - \underbrace{\arcsin 0}_{=0}\right) dx =$$

$$= 8a \arcsin\frac{b}{a}\int_0^a dx = 8a^2 \arcsin\frac{b}{a}$$ tražena površina

Zadaci za vježbu i rješenja

\# Izračunati površinu površi S, ako je S dio površi $z = \dfrac{x+y}{x^2+y^2}$ između cilindara $x^2+y^2=1$ i $x^2+y^2=2$ u I oktantu.

Rj.

Za površ $z = \dfrac{x+y}{x^2+y^2}$ imamo

$$\frac{\partial z}{\partial x} = \frac{y^2-2xy-x^2}{(x^2+y^2)^2}, \quad \frac{\partial z}{\partial y} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2},$$

pa je

$$P = \iint_S dS = \iint_D \sqrt{1+\left[\frac{y^2-2xy-x^2}{(x^2+y^2)^2}\right]^2+\left[\frac{x^2-2xy-y^2}{(x^2+y^2)^2}\right]^2}\,dxdy,$$

gdje je $D: 1 \leqslant x^2+y^2 \leqslant 2$.
Uvedimo polarne koordinate: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$. Imaćemo:

$$P = \iint_D \sqrt{1+\frac{1}{\rho^4}(-\cos 2\varphi-\sin 2\varphi)^2+\frac{1}{\rho^4}(\cos 2\varphi-\sin 2\varphi)^2}\,d\rho\, d\varphi =$$

$$= \int_0^{\pi/2} d\varphi \int_1^{\sqrt{2}} \frac{\sqrt{\rho^4+2}}{\rho^4}\cdot\rho^3\, d\rho = \frac{\pi}{2}\int_{\sqrt{3}}^{\sqrt{6}} \frac{t^2}{2(t^2-2)}\,dt =$$

$$\left(2+\rho^4=t^2, \quad \rho^3 d=\frac{1}{2}t\,dt\right)$$

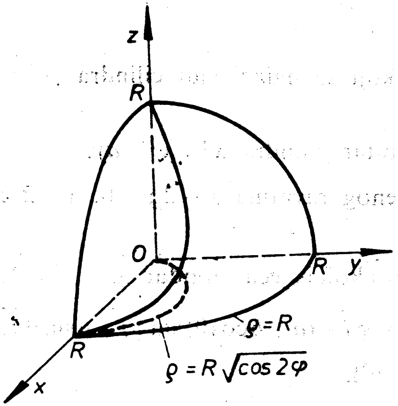
$$= \frac{\pi}{4}\int_{\sqrt{3}}^{\sqrt{6}}\left(1+\frac{2}{t^2-2}\right)dt = \frac{\pi}{4}\left(t+\frac{1}{\sqrt{2}}\ln\frac{t-\sqrt{2}}{t+\sqrt{2}}\right)\Bigg|_{\sqrt{3}}^{\sqrt{6}} = \cdots$$

$$= \frac{\pi}{4}\left[\sqrt{6}-\sqrt{3}-\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)\right.$$

\# Izračunati površinu površi S, ako je S dio površi $x^2+y^2+z^2=R^2$ koji se nalazi van cilindra $(x^2+y^2)^2=R^2(x^2-y^2)$.

Rj.

Površ $S: x^2+y^2+z^2=R^2$ isječena cilindrom $(x^2+y^2)^2=R^2(x^2-y^2)$ simetrična je odnosu na koordinatne ravni (sl. 70), pa je



Sl. 70

$$P=8\iint_{S_1} dS=$$

$$=8\iint_{S_1}\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dxdy=$$

$$=8\iint_{D_1}\frac{R}{\sqrt{R^2-(x^2+y^2)}}\,dxdy,$$

gdje je S_1 dio površi S u I oktantu. Uvedimo polarne koordinate. Biće

$$P=8R\int_{\pi/4}^{\pi/2}d\varphi\int_0^R\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}+8R\int_0^{\pi/4}d\varphi\int_{R\sqrt{\cos 2\varphi}}^{R}\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}.$$

(Može i ovako: $P=8R\int_0^{\pi/2}d\varphi\int_0^R\frac{\rho\,d\rho}{\sqrt{\rho^2-\rho^2}}-8R\int_0^{\pi/4}d\varphi\int_0^{R\sqrt{\cos\varphi}}\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}$.)

Dakle, $P=8R\int_0^{\pi/4}\left[-\sqrt{R^2-\rho^2}\right]_{R\sqrt{\cos 2\varphi}}^{R}d\varphi+8R\cdot\frac{\pi}{4}\left[-\sqrt{R^2-\rho^2}\right]_0^R.$

tj. $P=R^2(8\sqrt{2}-8+2\pi)$.

\# Izračunati površinu površi S, ako je:

250. S dio sfere $x^2+y^2+z^2=a^2$ unutar cilindra $x^2+y^2=ay$.

251. S dio cilindra $x^2=2z$ odsječenog ravnima $x-2y=0$, $y=2x$ i $x=2\sqrt{2}$.

252. S dio površi $y=x^2+z^2$ u I oktantu koji isijeca cilindar $x^2+z^2=1$.

253. S površ torusa $\vec{r}=(a+b\cos\theta)\cos\varphi\,\vec{i}+(a+b\cos\theta)\sin\varphi\,\vec{j}+b\sin\theta\,\vec{k}$.

Rj.

250. $P=4a^2\left(\dfrac{\pi}{2}-1\right)$.

251. $P=13$.

252. $P=\dfrac{(5\sqrt{5}-1)}{24}$.

253. $P=4ab\pi^2$ $\left(\text{Uzeti } P=\iint\limits_S dS=\iint\limits_D\left|\dfrac{\partial\vec{r}}{\partial\varphi}\times\dfrac{\partial\vec{r}}{\partial\theta}\right|d\varphi\,d\theta;\right.$

$\left.D:\ 0<\varphi\leqslant 2\pi,\ 0\leqslant\theta\leqslant 2\pi.\right)$.

\# Izračunati površinu površi S, ako je S površ $(x^2+y^2+z^2)^2 = a^2(x^2+y^2)$.

Rj.

Smjenom $x = \rho\cos\varphi\sin\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\theta$ dobija se jednačina $\rho = a\sin\theta$. Uvrštavajući ovu vrijednost $\rho = \rho(\varphi, \theta)$ u jednačine $x = x(\rho, \varphi, \theta)$, $y = y(\rho, \varphi, \theta)$, $z = z(\rho, \varphi, \theta)$ dobijaju se parametarske jednačine površi

$$x = a\sin^2\theta\cos\varphi = x(\varphi, \theta)$$

$$y = a\sin^2\theta\sin\varphi = y(\varphi, \theta)$$

$$z = \frac{a}{2}\sin 2\theta \quad = z(\varphi, \theta).$$

Za izračunavanje površine koristićemo vezu

$$\iint\limits_S dS = \iint\limits_D \sqrt{A^2+B^2+C^2}\,d\varphi\,d\theta, \text{ pri čemu je}$$

$$A = \frac{D(y, z)}{D(\varphi, \theta)}, \quad B = \frac{D(z, x)}{D(\varphi, \theta)}, \quad C = \frac{D(x, y)}{D(\varphi, \theta)}.$$

Biće:

$$A = a^2\sin^2\theta\cos\varphi\cos 2\theta$$

$$B = a^2\sin^2\theta\sin\varphi\cos 2\theta$$

$$C = -2a^2\cos\theta\sin^3\theta$$

i zatim

$$A^2+B^2+C^2 = a^4\sin^4\theta, \quad \sqrt{A^2+B^2+C^2} = a^2\sin^2\theta.$$

Sada je

$$P = a^2\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^2\theta\,d\theta = 2\pi a^2 \int_0^{\pi} \sin^2\theta\,d\theta = \pi^2 a^2.$$

\# Izračunati površinu površi S, ako je S površ (Vivanijevog) tijela

$$V=\{(x,\ y,\ z): x^2+y^2+z^2\leqslant R^2,\ x^2+y^2\leqslant Rx\}.$$

Rj.

Tijelo je simetrično u odnosu na ravan $z=0$, pa je $P=2P_S+2P_C$, pri čemu je P_S površina gornjeg dijela sfere, a P_C površina gornjeg dijela cilindra. Biće

$$P_S=\iint\limits_{S'}\sqrt{1+z_x'^2+z_y'^2}\,dxdy=R\iint\limits_{S'}\frac{dxdy}{\sqrt{R^2-(x^2+y^2)}}.$$

Oblast S' je krug $x^2+y^2\leqslant Rx$. Uvodeći polarne koordinate dobija se

$$P_S=R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{R\cos\varphi}\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}=2R^2\left(\frac{\pi}{2}-1\right).$$

Površinu cilindra tražimo pomoću krivolinijkog integrala: $P_C=\int\limits_l z\,ds$, pri čemu je l kružnica $x^2+y^2=Rx$, a $z=\sqrt{R^2-(x^2+y^2)}=\sqrt{R^2-Rx}.$

Ako jednačinu kružnice l napišemo u parametarskom obliku

$x=\frac{R}{2}(1-\cos\varphi)$, $y=\frac{R}{2}\sin\varphi$, dobiće se $ds=\frac{R}{2}\,d\varphi$, $\varphi\in(0,\ 2\pi)$, i zatim

$$P_C=\frac{R}{2}\int_0^{2\pi}\sqrt{R^2-\frac{R^2}{2}(1+\cos\varphi)}\,d\varphi=\frac{R^2}{2}\int_0^{2\pi}\sqrt{1-\cos^2\varphi}\,d\varphi=$$

$$=\frac{R^2}{2}\int_0^{2\pi}\sqrt{\sin^2\varphi}\,d\varphi=\frac{R^2}{2}\int_0^{2\pi}|\sin\varphi|\,d\varphi=$$

$$=\frac{R^2}{2}\int_0^{\pi}\sin\varphi\,d\varphi-\frac{R^2}{2}\int_{\pi}^{2\pi}\sin\varphi\,d\varphi=2R^2.$$

Slijedi

$$P=2R^2.$$

Stoksova formula

Dat je krivoliniski integral $\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ gdje je c kontura u prostoru.

Stoksova formula glasi:

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

površinski integral prve vrste

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

gdje je S površina u prostoru ograničena konturom c a $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na površinu S.

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma$$

Vidimo da Stoksova formula povezuje krivoliniski integral druge vrste sa površinskim integralom prve i druge vrste.

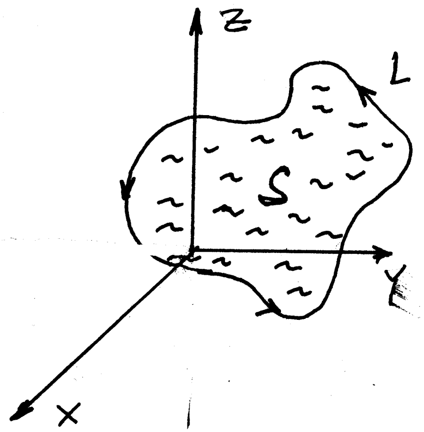
Ranije smo spomenuli Greenovu formulu koja povezuje krivoliniski integral druge vrste sa dvostrukim integralom.

Formula Gauss-Ostrogradski povezuje površinski integral druge vrste sa trostrukim integralom.

Integral $I=\int_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$

uzet po nekoj zatvorenoj konturi L, pretvoriti pomoću formule Stoksa u površinski integral, nad površinom koju zatvara spomenuta kontura.

Rj.



$$\int_L P\,dx+Q\,dy+R\,dz=\iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$R(x,y,z)=x^2+y^2$

$P(x,y,z)=y^2+z^2$

$Q(x,y,z)=x^2+z^2$

$\frac{\partial R}{\partial y}=2y \qquad \frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x \qquad \frac{\partial P}{\partial z}=2z$

$\frac{\partial Q}{\partial x}=2x \qquad \frac{\partial P}{\partial y}=2y$

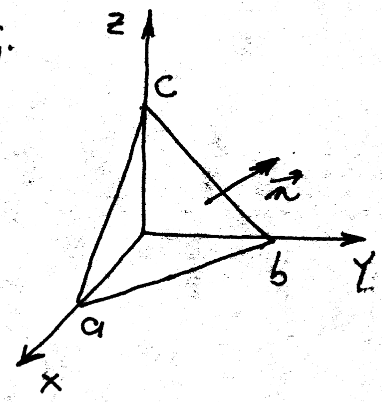
$$I=\iint_S (2y-2z)\,dy\,dz-(2x-2z)\,dx\,dz+(2x-2y)\,dx\,dy=$$

$$=2\iint_S (y-z)\,dy\,dz+(z-x)\,dx\,dz+(x-y)\,dx\,dy$$

(#) Izračunati krivoliniski integral $-\int\limits_{c} y^2dx+z^2dy+x^2dz$

pri čemu je c kontura ΔABC gde su tačke $A(a,0,0)$, $B(0,b,0)$ i $C(0,0,c)$, $a,b,c>0$.

Rj.



$$-\int\limits_{c} y^2dx+z^2dy+x^2dz \overset{\text{Stoksova formula}}{=\!=\!=} -\iint\limits_{S}\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R\end{vmatrix}$$

$P=y^2,\ Q=z^2,\ R=x^2$

$$\frac{\partial Q}{\partial x}=0,\quad \frac{\partial P}{\partial y}=2y,\quad \frac{\partial R}{\partial y}=0,\quad \frac{\partial Q}{\partial z}=2z$$

$$\frac{\partial R}{\partial x}=2x \qquad \frac{\partial P}{\partial z}=0$$

$$\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R\end{vmatrix} = -2z\,dydz-2x\,dzdx-2y\,dxdy$$

$$-\int\limits_{c} y^2dx+z^2dy+x^2dz = 2\iint\limits_{S}(z\,dydz+x\,dzdx+y\,dxdy)$$

S oblast ograničena ΔABC

Izračunajmo $\iint\limits_{S} z\,dydz$. Površinu S proicirajmo na yOz ravan:

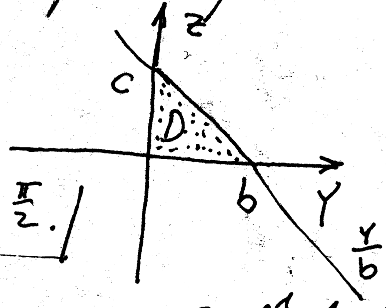
$$\frac{y}{b}+\frac{z}{c}=1$$

$cy+bz=bc$

$bz=bc-cy$

$$z=c-\frac{c}{b}y=\frac{c}{b}(b-y)$$

Ugao koji zaklapa vektor normale $\vec{n}$ na površ S je između 0 i $\frac{\pi}{2}$.



$$D:\begin{cases} 0\le y\le b \\ 0\le z\le \frac{c}{b}(b-y)\end{cases}$$

$\vec{n}=(\frac{1}{a},\frac{1}{b},\frac{1}{c})$

$\cos\alpha>0$

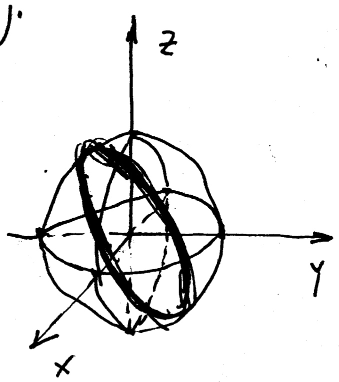
$$\iint\limits_{S} z\,dydz=+\iint\limits_{D} z\,dydz=\int\limits_0^b\Big[\int\limits_0^{\frac{c}{b}(b-y)} z\,dz\Big]dy=\int\limits_0^b \frac{1}{2}z^2\Big|_0^{\frac{c}{b}(b-y)}dy=\frac{1}{2}\Big(\frac{c}{b}\Big)^2\int\limits_0^b(b-y)^2dy$$

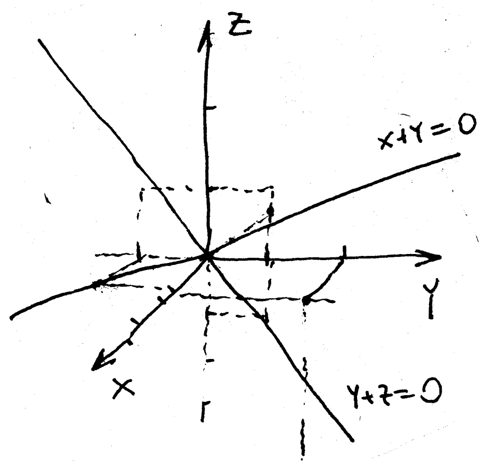
$$=\left|\begin{matrix} b-y=t & -dy=dt \\ y=0 \Rightarrow t=b & dy=-dt \\ y=b \Rightarrow t=0 & \end{matrix}\right|=\frac{1}{2}\frac{c^2}{b^2}\int\limits_0^b t^2dt=\frac{1}{2}\cdot\frac{c^2}{b^2}\frac{t^3}{3}\Big|_0^b=\frac{1}{2}\cdot\frac{bc^2}{3}$$

Analogno izračunamo $\iint\limits_{S} x\,dzdx=\frac{1}{2}\,\frac{a^2c}{3}$ i $\iint\limits_{S} y\,dxdy=\frac{1}{2}\cdot\frac{ab^2}{3}$ $\Rightarrow$ $I=\frac{ab^2+bc^2+ca^2}{3}$

(#) Izračunati krivoliniski integral $\int y\,dx + z\,dy + x\,dz$ ako je c krug dobijen presjekom c sfere $x^2+y^2+z^2=a^2$ i ravni $x+y+z=0$.

Rj.





$$\int_C y\,dx + z\,dy + x\,dz \xlongequal{\text{Stoksova formula}} \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$\frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = 1 \qquad \begin{matrix} P = y \\ Q = z \\ R = x \end{matrix}$$

S je površina ograničena krugom

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma$$

$$\frac{\partial R}{\partial y} = 0 \quad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial x} = 1 \quad \frac{\partial P}{\partial z} = 0$$

$$\int_C y\,dx + z\,dy + x\,dz = \iint_S (-\cos\alpha - \cos\beta - \cos\gamma)\,dS$$

gdje je $\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$ vektor (jedinični) normale na površinu S

$x+y+z=0$

$\vec{n} = (1,1,1)$ vektor normale na ravan $x+y+z=0$ (a time i na našu površinu S)

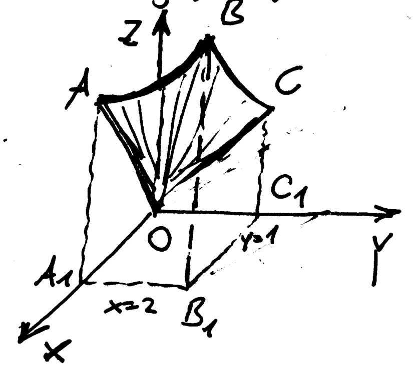
$$|\vec{n}| = \sqrt{1+1+1} = \sqrt{3} \qquad \vec{n}_0 = \Big(\underbrace{\frac{1}{\sqrt{3}}}_{\cos\alpha}, \underbrace{\frac{1}{\sqrt{3}}}_{\cos\beta}, \underbrace{\frac{1}{\sqrt{3}}}_{\cos\gamma}\Big)$$

$$\iint_S (-\cos\alpha - \cos\beta - \cos\gamma)\,dS = \iint_S \left(-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) dS = -\frac{3}{\sqrt{3}} \iint_S dS$$

$\iint_S dS$ je površina oblasti S (S je krug poluprečnika a $P_{kruga} = a^2\pi$)

$$\int_C y\,dx + z\,dy + x\,dz = -\frac{3}{\sqrt{3}} a^2\pi = -\sqrt{3}\,a^2\pi$$

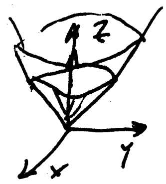
(#) Uz pomoć formule Stoksa, izračunati krivoliniski integral $K=\oint_{\ell} e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$ gdje je



ℓ - zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina

$z=\sqrt{x^2+y^2}$, $x=0$, $x=2$, $y=0$, $y=1$.

Rj.: $z=\sqrt{x^2+y^2}$ je čunj iznad xOy ravni



$x=0$, $x=2$ su ravni paralelne sa yOz ravni

$y=0$, $y=1$ su ravni paralelne sa xOz ravni

Stoksova formula glasi

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$P=e^x$, $\frac{\partial P}{\partial y}=0$, $\frac{\partial P}{\partial z}=0$

$Q=z(x^2+y^2)^{\frac{3}{2}}$, $\frac{\partial Q}{\partial x} = z\cdot\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\cdot 2x = 3xz\sqrt{x^2+y^2}$, $\frac{\partial Q}{\partial z}=(x^2+y^2)^{\frac{3}{2}}$

$R=yz^3$, $\frac{\partial R}{\partial x}=0$, $\frac{\partial R}{\partial y}=z^3$.

$$K=\oint_{\ell} e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz = \begin{vmatrix} \text{formula Stoksa} \end{vmatrix} =$$

$$= \iint_S \begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & z(x^2+y^2)^{\frac{3}{2}} & yz^3 \end{vmatrix} = \iint_S \underbrace{(z^3 - \underset{(\sqrt{x^2+y^2})^3}{\underset{\parallel}{(x^2+y^2)^{\frac{3}{2}}}})}_{=0} dydz - (0-0)dxdz$$

$$+ (3xz\sqrt{x^2+y^2} - 0)\,dxdy = \iint_S 3xz\sqrt{x^2+y^2}\,dxdy$$

površinski integral II vrste

Tj. dobili smo $K=\iint\limits_S 3xz\sqrt{x^2+y^2}\,dx\,dy$

Kako naša data kriva pravi površinu $S: z=\sqrt{x^2+y^2}$ u prvom oktantu imamo

$$K=\iint\limits_S 3x\,(x^2+y^2)\,dx\,dy$$

Prisjetimo se kako se računa površinski integral II vrste npr. $I=\iint\limits_S R(x,y,z)\,dx\,dy$. Neka je $\vec{n}$ vektor normale na površ S, i neka je γ ugao koji $\vec{n}$ gradi sa z-osom, i neka je D projekcija površi S na xOy ravan. Tada

$I=\iint\limits_S R(x,y,z)\,dx\,dy=\pm\iint\limits_D R(x,y,z(x,y))\,dx\,dy$ gdje predznak ispred integrala zavisi od $\cos\gamma$ (za $\cos\gamma>0$ $+$, $\cos\gamma<0$ $-$).

Mi posmatramo vanjsku stranu površi, iz čega možemo zaključiti (sa slike) da je $\gamma\in(\frac{\pi}{2},\pi)$ pa je $\cos\gamma<0$.
Projekcija D površi S je data u sklopu zadatka (vidi sliku) ($\square A_1B_1C_1O$)

$$D:\begin{cases}0\le x\le 2\\ 0\le y\le 1\end{cases}\qquad K=\iint\limits_S 3x(x^2+y^2)\,dx\,dy=-\iint\limits_D 3x(x^2+y^2)\,dx\,dy=$$

$$=-3\int_0^1 dy\int_0^2 (x^3+xy^2)\,dx=-3\int_0^1\left(\frac{1}{4}x^4\Big|_0^2+\frac{1}{2}x^2y^2\Big|_0^2\right)dy=$$

$$=-3\int_0^1(4+2y^2)\,dy=-3\left(4y\Big|_0^1+\frac{2}{3}y^3\Big|_0^1\right)=-12-2=-14$$ traženo rješenje

Zadaci za vježbu

3894. Integral $\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$, uzet po nekoj zatvorenoj konturi L, primenom Štoksove formule transformisati u integral po površini „razapetoj" nad tom konturom.

3895. Izračunati integral $\int\limits_L x^2 y^3\,dx+dy+z\,dz$ po krugu $x^2+y^2=R^2$, $z=0$, na dva načina: a) neposredno, i b) koristeći Štoksovu formulu, uzimajući za površinu S polusferu $z=+\sqrt{R^2-x^2-y^2}$. (Integracija po krugu u ravni xOy računa se u pozitivnom smeru obilaženja).

Rješenja

3894. $2\iint\limits_S (x-y)\,dx\,dy+(y-z)\,dy\,dz+(z-x)\,dx\,dz.$

3895. $-\dfrac{\pi R^6}{8}.$

Formula Gauss-Ostrogradski

Ova formula daje vezu između površinskog integrala druge vrste i trostrukog integrala.

$$\iint P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S (S je zatvorena površina).

1. Izračunati $\iint\limits_{S} xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$ gdje je S bilo koja zatvorena površ.

Rj.

$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iint\limits_{S} P\,dy\,dz + Q\,dx\,dz +$$

$$+ R\,dx\,dy \overset{\text{Formula Gauss-Ostr.}}{=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S. $\frac{\partial P}{\partial x} = 0$, $\frac{\partial Q}{\partial y} = 0$, $\frac{\partial R}{\partial z} = 0$

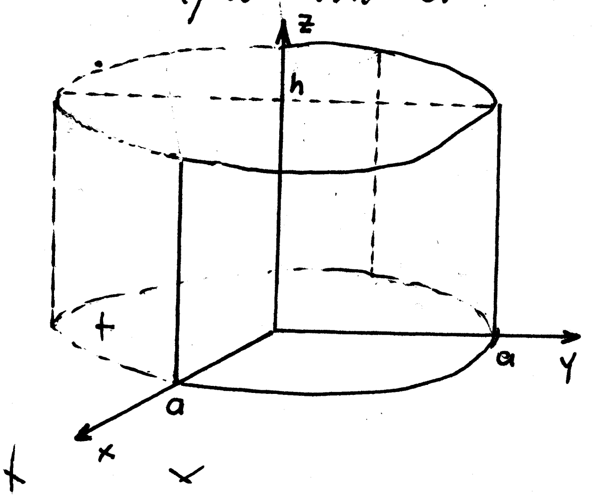
$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iiint\limits_{\Omega} 0\,dx\,dy\,dz =$$

$$\Omega : \begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases} \qquad = \int_a^b dx \int_c^d dy \int_e^f 0\,dz = 0$$

(#) Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral $I=\oiint\limits_S 4x^3\,dy\,dz+4y^3\,dx\,dz-6z^4\,dx\,dy$

gdje je S vanjska strana cilindra $x^2+y^2=a^2$ koji se nalazi između ravni $z=0$ i $z=h$.

Rj. Skicirajmo dati cilindar



Prisjetimo se formule Gauss-Ostrogradski

$$\oiint\limits_S P(x,y,z)\,dy\,dz+Q(x,y,z)\,dx\,dz+R(x,y,z)\,dx\,dy=$$

$$=\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

Ω - unutrašnjost oblasti S

$P(x,y,z)=4x^3 \qquad \frac{\partial P}{\partial x}=12x^2$

$Q(x,y,z)=4y^3 \qquad \frac{\partial Q}{\partial y}=12y^2$

$R(x,y,z)=6z^4 \qquad \frac{\partial R}{\partial z}=24z^3$

$$\oiint\limits_S 4x^3\,dy\,dz+4y^3\,dx\,dz-6z^4\,dx\,dy=12\iiint\limits_{\Omega}(x^2+y^2-2z^3)\,dx\,dy\,dz=$$

$$=\left|\begin{array}{ll} \text{uvedimo cilindrične koordinate} & \\ x=r\cos\varphi & \Omega \xrightarrow{\text{transformiše}} \Omega' \\ y=r\sin\varphi & 0\le r\le a \\ z=z & 0\le\varphi\le 2\pi \\ dx\,dy\,dz=r\,dr\,d\varphi\,dz & 0\le z\le h \\ x^2+y^2=\ldots=r^2 & \end{array}\right| = 12\iiint\limits_{\Omega'}(r^2-2z^3)\,r\,dr\,d\varphi\,dz=$$

$$=12\int_0^{2\pi}d\varphi\int_0^a dr\int_0^h(r^3-2rz^3)\,dz=12\int_0^{2\pi}d\varphi\int_0^a\left(r^3z\Big|_0^h-2r\cdot\frac{1}{4}z^4\Big|_0^h\right)dr$$

$$=12\int_0^{2\pi}d\varphi\int_0^a\left(r^3h-\frac{1}{2}rh^4\right)dr=12\,\varphi\Big|_0^{2\pi}\left(h\frac{1}{4}r^4\Big|_0^a-\frac{1}{2}h^4\cdot\frac{1}{2}r^2\Big|_0^a\right)=$$

$$=24\pi\cdot\frac{1}{4}h\left(a^4-h^3a^2\right)=6\pi ha^2(a^2-h^3) \quad \text{traženo rješenje}$$

(#) Površinski integral po zatvorenoj površini pretvoriti uz pomoć formule Ostrogradskog u trostruki integral po zapremini tijela, koje je ograničeno spomenutom površinom

$$\iint_S \sqrt{x^2+y^2+z^2}\,[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS$$

gdje je $\vec{n}$ vanjska normala na površinu S.

Rj: $\cos(\vec{n},x)$ je kosinus ugla između normale i x-ose.
$\cos(\vec{n},y)$ i $\cos(\vec{n},z)$ je kosinus ugla između normale na površinu S i y-ose i z-ose redom.

Uvedimo oznake $\cos(\vec{n},x)=\cos\alpha$, $\cos(\vec{n},y)=\cos\beta$ i $\cos(\vec{n},z)=\cos\gamma$.

Prema formuli Stoksa znamo da je
$dy\,dz = dS\cos\alpha$
$dz\,dx = dS\cos\beta$
$dx\,dy = dS\cos\gamma$

$$I=\iint_S \sqrt{x^2+y^2+z^2}\,(\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS =$$

$$=\iint_S \sqrt{x^2+y^2+z^2}\,(dy\,dz+dz\,dx+dx\,dy)$$

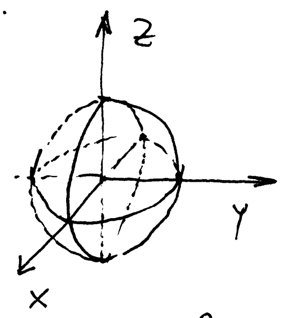
$$\iint_S P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\partial P}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}} \qquad I=\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

(#) Izračunati $\iint\limits_S x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy$ gdje je

S - vanjski dio sfere $x^2+y^2+z^2=R^2$.

Rj:



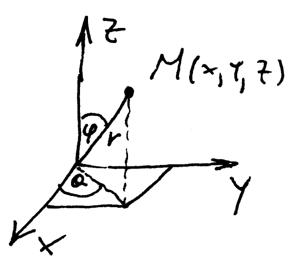
$$I=\iint\limits_S x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy \overset{\text{formula Gauss-Ostrogradski}}{=\!=}$$

$$=\iiint\limits_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$P=x^3,\quad \frac{\partial P}{\partial x}=3x^2,\quad Q=y^3,\quad \frac{\partial Q}{\partial y}=3y^2,\quad R=z^3,\quad \frac{\partial R}{\partial z}=3z^2$$

$$I=\iiint\limits_\Omega (3x^2+3y^2+3z^2)\,dx\,dy\,dz \qquad \Omega: x^2+y^2+z^2\le R^2$$

Uvedimo sferne koordinate:



$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

$$\Omega': \begin{cases} 0\le r\le R \\ 0\le\theta\le 2\pi \\ 0\le\varphi\le\pi \\ dx\,dy\,dz=r^2\sin\varphi\,d\varphi\,d\theta\,dr \end{cases}$$

$$x^2+y^2+z^2=r^2$$

$$I=3\iiint\limits_{\Omega'} r^2 r^2\sin\varphi\,d\varphi\,d\theta\,dr=3\int_0^R r^4\,dr\int_0^{2\pi}d\theta\int_0^\pi \sin\varphi\,d\varphi=$$

$$=3\cdot\frac{1}{5}r^5\Big|_0^R\cdot\theta\Big|_0^{2\pi}\cdot(-\cos\varphi)\Big|_0^\pi=\frac{3}{5}\cdot R^5\cdot 2\pi\cdot 2=\frac{12}{5}R^5\pi$$

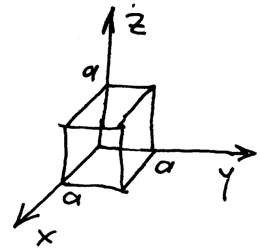
(#) Izračunati $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$ gdje je

S - vanjska strana kocke $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Rj.

$$\iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \overset{\text{formula Gauss-Ostr.}}{=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial Q}{\partial y} = 2y, \quad \frac{\partial R}{\partial z} = 2z$$



$$\Omega : \begin{cases} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{cases}$$

Prema tome:

$$\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy =$$

$$= \iiint\limits_{\Omega} (2x + 2y + 2z)\,dx\,dy\,dz = 2\int_0^a dx \int_0^a dy \int_0^a (x+y+z)\,dz =$$

$$= 2\int_0^a dx \int_0^a \left(xz\Big|_0^a + yz\Big|_0^a + \frac{1}{2}z^2\Big|_0^a \right) dy = 2\int_0^a dx \int_0^a \left(ax + ay + \frac{1}{2}a^2\right) dy =$$

$$2a\int_0^a \left(xy\Big|_0^a + \frac{1}{2}y^2\Big|_0^a + \frac{1}{2}ay\Big|_0^a \right) dx = 2a\int_0^a \left(ax + \frac{1}{2}a^2 + \frac{1}{2}a^2\right) dx = 2a^2\int_0^a (x+a)\,dx =$$

$$= 2a^2\left(\frac{1}{2}a^2 + a^2\right) = 3a^4$$

Zadaci za vježbu

3896. Površinski integral $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$, uzet po zatvorenoj površini S, primenom formule Ostrogradskog, transformisati u trojni integral po zapremini ograničenoj tom površinom (Integral se računa po spoljnoj strani površine S).

3897. Površinski integral $\iint\limits_S x^2 + y^2 + z^2 \{\cos(N, x) + \cos(N, y) + \cos(N, z)\} d\sigma$ po zatvorenoj površini S, primenom formule Ostrogradskog transformisati u trojni integral po zapremini ograničenoj tom površinom, pri čemu je N spoljna normala površine S.

3898. Izračunati integral u prethodnom zadatku ako je S sfera poluprečnika R sa centrom u koordinatnom početku.

3899. Izračunati integral

$$\iint\limits_S [x^3 \cos(N,\ x) + y^3 \cos(N,\ y) + z^3 \cos(N,\ z)]\, d\sigma,$$

u kojem je S — sfera poluprečnika R sa centrom u koordinatnom početku, a N — spoljna normala.

3900. Izračunati integral u zadacima 3891—3863 primenom formule Ostrogradskog.

Rješenja

3896. $2\iiint\limits_\Omega (x + y + z)\,dx\,dy\,dz.$

3897. $\iiint\limits_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz.$ **3898.** 0. **3899.** $\frac{12}{5}\pi R^5.$